Pergamon

Int. J. Solids Structures Vol. 33, No. 25, pp. 3697–3721, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00207–3

CORNERS IN PLASTICITY—KOITER'S THEORY REVISITED

NIELS SAABYE OTTOSEN and MATTI RISTINMAA
Department of Solid Mechanics, University of Lund, Box 118, S-221 00 Lund, Sweden

(*Received* 24 *January* 1995; *in revised form* 21 *August* 1995)

Abstract—A general theory for plastic loading at corners is presented that includes Koiter's theory as a special case. This theory is derived within a thermodynamic framework and includes non-associated as well as associated theory. The non-associated theory even allows the number of potential functions to differ from the number of yield functions. The properties of the matrix of plastic moduli as well as of another important matrix are discussed in detail and hardening, perfect and softening plasticity are concisely defined. The existence of limit points is also discussed.

The strain driven format turns out to be the most general. Moreover, consistent loading and unloading criteria are established for general non-associated plasticity. An explicit criterion for uniqueness is derived, and finally, some of the general findings are illustrated by means of specific plasticity formulations often encountered in practice. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

According to Drucker's postulate (1951), the direction of the plastic strain rate $\dot{\varepsilon}_{ij}^p$ for associated plasticity is bounded as illustrated in Fig. 1.

In agreement with that which Koiter (1953, 1960) proposed for perfect associated plasticity, the flow rule at a corner is said to be

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}^I \frac{\partial f^I}{\partial \sigma_{ij}} \quad I = 1, 2, \ldots, F_{max} \tag{1}$$

in which f^I are yield functions and the summation convention is also adopted for capital letters used as superscripts. In (1), F_{max} is the total number of yield surfaces that meet at a corner. Moreover, the plastic multipliers $\dot{\lambda}^I$ are determined by the consistency relation and we have

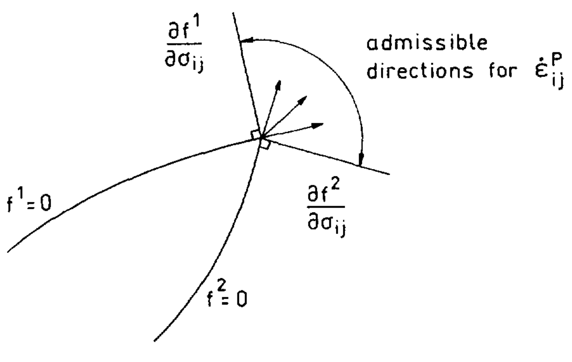

Fig. 1. Stress space with two yield functions. Admissible directions for $\dot{\varepsilon}_{ij}^p$ at a corner according to Drucker's postulate.

$$\dot{\lambda}^I = 0 \quad \text{if} \quad \begin{cases} f^I < 0 \\ f^I = 0 \quad \text{and} \quad \dfrac{\partial f^I}{\partial \sigma_{ij}} \dot{\sigma}_{ij} < 0 \end{cases}$$

$$\dot{\lambda}^I > 0 \quad \text{if} \quad f^I = 0 \quad \text{and} \quad \frac{\partial f^I}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = 0 \tag{2}$$

For hardening associated plasticity, Koiter (1953, 1960) proposed

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda}^I \frac{\partial f^I}{\partial \sigma_{ij}} \tag{3}$$

where the plastic multipliers $\dot{\lambda}^I$ were given by

$$\dot{\lambda}^I = C^{(I)} h^{(I)} \frac{\partial f^{(I)}}{\partial \sigma_{kl}} \dot{\sigma}_{kl} \tag{4}$$

and the summation convention is *not* applied to capital letters in parentheses. Moreover, in (4) we have

$$C^{(I)} = 0 \quad \text{if} \quad \begin{cases} f^{(I)} < 0 \\ f^{(I)} = 0 \quad \text{and} \quad \dfrac{\partial f^{(I)}}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \leqslant 0 \end{cases} \tag{5}$$

$$C^{(I)} = 1 \quad \text{if} \quad f^{(I)} = 0 \quad \text{and} \quad \frac{\partial f^{(I)}}{\partial \sigma_{ij}} \dot{\sigma}_{ij} > 0.$$

In addition, $h^{(I)}$ are given positive functions of the stress tensor and the plastic history. Three points may be raised against Koiter's theory for hardening associated plasticity.

First, in the limit approaching perfect plasticity, we will have $\dot{\sigma}_{ij}\,\partial f^{(I)}/\partial \sigma_{ij} \to 0$ which, in order that $\dot{\lambda}^I$ be a finite quantity different from zero, requires that $h^{(I)} \to \infty$. In turn, this implies that perfect plasticity cannot be obtained as a limit case of the formulation (3) and (4). Moreover, the conditions for obtaining softening plasticity were not given.

Second, the formulation (4) assumes, *a priori*, that the plastic multipliers can be expressed in terms of the stress rate $\dot{\sigma}_{ij}$, i.e. the formulation (3) and (4) assumes, *a priori*, that the response can be determined if the loading is prescribed in terms of a given stress history.

Third, even if we accept a stress driven format, it turns out that (3) and (4) are not the most general associated formulations. It may be noted that in the Koiter theory, the yield function $f^{(I)}$ only influences the corresponding plastic multiplier $\dot{\lambda}^{(I)}$; this is termed independent hardening. Staying within hardening associated plasticity, Mandel (1965) generalized this to include so-called dependent hardening, where more yield functions influence the same multiplier $\dot{\lambda}^{(I)}$. However, both Koiter's (1953, 1960) theory and other later contributions, (for example, Mandel (1965), Hill (1966), Sewell (1973, 1974), Moreau (1974) and Simo *et al.* (1988)) restrict the formulation to associated plasticity. Apart from these remarks, Koiter showed that if the formulations (3) and (4) are accepted then the response is uniquely determined once the loading is given.

It is of interest that, from a historical point of view, the stress driven format was accepted as the most natural one and it was even questioned, cf. Warner and Handelman (1956), whether a strain driven format could be derived. As will be shown, it is, in fact, the establishment of the stress driven format that puts the severest restrictions on a proper formulation.

In this paper, we shall derive the evolution laws for general non-associated and associated plasticity when corners of the yield surface and/or potential surface exist. As an important generalization, we shall allow the number of potential functions to differ from the number of yield functions. This derivation is based upon a formulation that fulfills all thermodynamical requirements. We shall then present a discussion of the general properties of the so-called matrix of plastic moduli as well as of another important matrix. This discussion leads to concise definitions of hardening, perfect and softening plasticity; also the existence of limit points is evaluated.

Even for associated plasticity, proper loading/unloading criteria are not trivial, as discussed by Mandel (1965), Sewell (1973) and Simo *et al.* (1988). For general non-associated plasticity, where the number of potential and yield functions do not need to be the same, the situation becomes even more complex and we present the relevant loading/unloading criteria.

The strain driven format turns out to be the natural formulation and the conditions for deriving a stress driven format are discussed. The conditions for which uniqueness exists are evaluated in detail, and finally, the general findings are illustrated by means of plasticity formulations often encountered in practice.

2. THERMODYNAMIC BASIS

The assumption of small strains is made. This allows for a decomposition of the total strain tensor into an elastic part ε_{ij}^e and a plastic part ε_{ij}^p, i.e.

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \tag{6}$$

For isothermal conditions, let us consider the following form of Helmholtz's free energy function ψ per unit volume

$$\psi(\varepsilon_{ij}^e, \kappa_\alpha) = \psi^e(\varepsilon_{ij}^e) + \psi^p(\kappa_\alpha) \tag{7}$$

where κ_α denotes a set of plastic variables or internal variables which may be scalars or second-order tensors. Moreover, the number of internal variables may be one, two or more and this is indicated by the Greek subscript α. The decomposition (7) corresponds to the assumption that the instantaneous elastic response does not depend on the internal variables κ_α, cf. Lubliner (1972). Since isothermal conditions are assumed, the second law of thermodynamics, i.e. the Clausius-Duhem inequality, takes the form

$$-\dot{\psi} + \sigma_{ij}\dot{\varepsilon}_{ij} \geqslant 0 \tag{8}$$

for any admissible process. Taking the time derivative of (7), substituting into (8) and making use of (6), we obtain that an allowable solution is given by

$$\sigma_{ij} = \frac{\partial \psi^e}{\partial \varepsilon_{ij}^e} \tag{9}$$

and the dissipation inequality

$$\dot{D} = \sigma_{ij}\dot{\varepsilon}_{ij}^p + A_\alpha \dot{\kappa}_\alpha \geqslant 0 \tag{10}$$

where we have defined the thermodynamic conjugated forces A_α by

$$A_\alpha = -\frac{\partial \psi^p}{\partial \kappa_\alpha}. \tag{11}$$

Differentiation of (9) with respect to time along with the decomposition (6) then yields Hooke's law

$$\dot{\sigma}_{ij} = D_{ijkl}(\dot{\varepsilon}_{kl} - \dot{\varepsilon}^p_{kl}) \tag{12}$$

where

$$D_{ijkl} = \frac{\partial^2 \psi^e}{\partial \varepsilon^e_{ij}\, \partial \varepsilon^e_{kl}}. \tag{13}$$

We note that the elastic stiffness tensor D_{ijkl} is symmetric and in general not constant. However, we shall assume D_{ijkl} to be positive definite.

3. INTERNAL VARIABLES

From the thermodynamic formulation, the laws for the stress tensor (9) and the thermodynamic forces (11) were obtained, but no information is given about the evolution laws for the plastic strains and the internal variables. The only restriction on these evolution laws is that the second law of thermodynamics be fulfilled, i.e., that the dissipation inequality (10) must be fulfilled.

To obtain these evolution laws, a set of potential functions is introduced with the form

$$g^\Phi = g^\Phi(\sigma_{ij}, A_\alpha) \quad \Phi = 1, 2, \ldots, G_{max} \tag{14}$$

where G_{max} is the total number of potential surfaces that meet at a corner. It is emphasized that whereas the full set of conjugated forces is given by A_1, $A_2, \ldots, A_\alpha$, the formulation (14) allows for the possibility that only some of the forces A_α enter the expression for, say g^1, whereas other forces A_α enter the expression for, say, g^4. However, regardless of conjugated forces that enter the expression for a specific potential function, these forces belong to the full set of all conjugated forces A_α.

The potential functions are assumed to be smooth and convex functions. In the σ_{ij}, A_α- space, $g^\Phi = 0$ describes for each Φ-value a surface in that space. These potential functions g^Φ are chosen such that $g^\Phi(\sigma_{ij} = 0, A_\alpha = 0) < 0$. The following evolution laws are now postulated

$$\left.\begin{aligned} \dot{\varepsilon}^p_{ij} &= \dot{\lambda}^\Phi \frac{\partial g^\Phi}{\partial \sigma_{ij}} \\ \dot{\kappa}_\alpha &= \dot{\lambda}^\Phi \frac{\partial g^\Phi}{\partial A_\alpha} \end{aligned}\right. \qquad \dot{\lambda}^\Phi \geqslant 0. \tag{15}$$

With these assumptions it appears that the evolution laws fulfill the dissipation inequality (10). The evolution laws (15) hold for hardening, perfect and softening plasticity. We may also note that the flow rule in (15), i.e., the evolution law for $\dot{\varepsilon}^p_{ij}$, is a direct generalization of the corresponding flow rule for a smooth potential function.

Similar to (14) the yield functions are assumed to be of the form

$$f^I = f^I(\sigma_{ij}, A_\alpha) \quad I = 1, 2, \ldots, F_{max} \tag{16}$$

The same interpretation given above for the potential functions also holds for the conjugated forces that enter a specific yield function. For any process we have the constraint conditions

$$f^I \leqslant 0 \tag{17}$$

and development of plastic effects requires that $f^I = 0$. It is noted that we do not require that the number of potential functions be equal to the number of yield functions, since, in general, we have $G_{max} \neq F_{max}$, and we shall return to this aspect later on.

If instead we adopt the postulate of maximum dissipation, we are faced with the following problem: for given $\dot{\varepsilon}^p_{ij}$ and $\dot{\kappa}_\alpha$, minimize the quantity $-\dot{D}$, where $\dot{D}$ is given by (10), under the constraints expressed by (17). Following, for instance, Luenberger (1984) p. 314, we are then led to

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}^I \frac{\partial f^I}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \dot{\lambda}^I \frac{\partial f^I}{\partial A_\alpha} \tag{18}$$

with the Kuhn-Tucker relations given by

$$\dot{\lambda}^I \geqslant 0 \quad \text{and} \quad \dot{\lambda}^{(I)} f^{(I)} = 0 \quad \text{no summation over } I \tag{19}$$

where the normality rule for the plastic strain rate is illustrated in Fig. 1. A prerequisite for (18) and (19) is that the point, i.e., the state in question, is a regular point. This means that the quantities $\partial f^I/\partial \sigma_{ij}$ are linearly independent; likewise $\partial f^I/\partial A_\alpha$ are linearly independent. Moreover, in order to fulfill the postulate of maximum dissipation, it is required that f^I be convex functions.

It appears readily that expressions (18) and (19) imply associated plasticity. The non-associated formulation (15) reduces to (18) and (19) for $G_{max} = F_{max}$ and $f^I = g^I$. Moreover, in order that the non-associated formulation should contain associated plasticity as a special case, it seems natural to require that

$$\begin{aligned} &\frac{\partial f^I}{\partial \sigma_{ij}} \text{ are linearly independent} \\ &\frac{\partial g^\Phi}{\partial \sigma_{ij}} \text{ are linearly independent} \\ &\frac{\partial g^\Phi}{\partial A_\alpha} \text{ are linearly independent} \end{aligned} \tag{20}$$

as well as f^I and g^Φ be convex functions.

4. PLASTIC MODULI MATRIX AND STRAIN DRIVEN FORMAT

From (16), we obtain

$$\dot{f}^I = \frac{\partial f^I}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f^I}{\partial A_\alpha} \dot{A}_\alpha.$$

Use of (11) in this relation implies

$$\dot{f}^I = \frac{\partial f^I}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \frac{\partial f^I}{\partial A_\alpha} \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta} \dot{\kappa}_\beta.$$

With the general evolution laws (15), we then obtain

$$\dot{f}^I = \frac{\partial f^I}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - H^{I\Phi} \dot{\lambda}^\Phi \tag{21}$$

where the matrix of plastic moduli $H^{I\Phi}$ is defined by

$$H^{I\Phi} = \frac{\partial f^I}{\partial A_\alpha} \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta} \frac{\partial g^\Phi}{\partial A_\beta}. \tag{22}$$

Insertion of the flow rule (15) into the Hooke's law (12) yields

$$\dot{\sigma}_{ij} = D_{ijkl} \dot{\varepsilon}_{kl} - D_{ijst} \frac{\partial g^\Phi}{\partial \sigma_{st}} \dot{\lambda}^\Phi. \tag{23}$$

With this expression, (21) takes the form

$$\dot{f}^I = \dot{a}^I - A^{I\Phi} \dot{\lambda}^\Phi \quad \text{where} \quad \dot{a}^I = \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl} \dot{\varepsilon}_{kl} \tag{24}$$

where the matrix $A^{I\Phi}$ is defined by

$$A^{I\Phi} = H^{I\Phi} + \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial g^\Phi}{\partial \sigma_{kl}}. \tag{25}$$

Later we shall discuss the loading/unloading criteria which, in general, turn out to be quite complex. However, for the moment we assume that no elastic unloading occurs. In that case, the consistency relations $\dot{f}^I = 0$ become in combination with (24)

$$A^{I\Phi} \dot{\lambda}^\Phi = \dot{a}^I. \tag{26}$$

In order to be able to derive a strain driven format, it is necessary that this non-homogeneous equation system allow for a unique $\dot{\lambda}^\Phi$-solution. This requirement turns out to place restrictions on the character of the matrix $A^{I\Phi}$.

The augmented matrix $\mathbf{T}$ is defined by

$$\mathbf{T} = [A^{I\Phi}, \dot{a}^I] \quad (F_{max} \times (G_{max} + 1))$$

where the coefficient matrix $A^{I\Phi}$ has the dimension $(F_{max} \times G_{max})$. A unique solution of (26), where G_{max} is the number of unknowns, requires that $\text{Rank}(\mathbf{T}) = \text{Rank}(\mathbf{A}) = G_{max}$, i.e.,

$$\text{the rank of } A^{I\Phi} = G_{max} \tag{27}$$

and

$$\dot{a}^I \quad \text{can be expressed by linear combinations of the columns in } A^{I\Phi}. \tag{28}$$

If $F_{max} < G_{max}$, requirement (27) can never be fulfilled, i.e.,

$$F_{max} \geqslant G_{max}. \tag{29}$$

Therefore, the number of yield functions must be larger than or equal to the number of potential functions. Even if (29) is fulfilled, the matrix $A^{I\Phi}$ must, in addition, fulfill (27). If $F_{max} = G_{max}$ and (27) holds, we observe that requirement (28) is fulfilled trivially.

Due to (27), there exists a left inverse $\underline{A}^{\Theta I}$ such that

$$\underline{A}^{\Theta I} A^{I\Phi} = \delta^{\Theta\Phi} \tag{30}$$

where $\delta^{\Theta\Phi}$ denotes a generalized Kronecker delta, cf. for example, Ayres (1962) p.63. Since $A^{I\Phi}$ has the dimension $(F_{max} \times G_{max})$, $\underline{A}^{\Theta I}$ has the dimension $(G_{max} \times F_{max})$.

The requirements on $A^{I\Phi}$ even need to be strengthened. Since (27) and (29) must be fulfilled, it is always possible to choose the numbering of the yield functions such that $A^{I\Phi}$ can be written in the format

$$A^{I\Phi} = \begin{bmatrix} P^{\Theta\Phi} \\ Q^{Y\Phi} \end{bmatrix}; \quad \text{Rank}\, P^{\Theta\Phi} = G_{max} \tag{31}$$

where $P^{\Theta\Phi}$ has the dimension $(G_{max} \times G_{max})$ and $Q^{Y\Phi}$ has the dimension $((F_{max} - G_{max}) \times G_{max})$. Considering as a special case that of perfect plasticity, i.e., no conjugated forces A_α exist in the yield functions, then $H^{I\Phi}$ defined by (22) becomes the zero matrix. If, in addition, we assume associated plasticity then $A^{I\Phi}$ reduces to the symmetric square matrix $A^{I\Phi} = A^{IJ}$ where

$$A^{IJ} = \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial f^J}{\partial \sigma_{kl}}$$

since D_{ijkl} is positive definite, it follows that also A^{IJ} is positive definite, i.e., A^{IJ} possesses positive eigenvalues.* As the theory that we want to develop should be general, we therefore assume—in recognition of (31)—that

$$\text{the eigenvalues of } P^{\Theta\Phi} \text{ should be positive.} \tag{32}$$

This requirement is assumed to hold in general. Considering the special case of associated plasticity where $F_{max} = G_{max}$ and A^{IJ} is symmetric, Sewell (1973) and Simo *et al.* (1988) argue that A^{IJ} should be positive definite. For $F_{max} = G_{max}$ and associated plasticity, (32) reduces to that statement.

Having identified the necessary properties of $A^{I\Phi}$, we return to the solution of (26). Multiplication of (26) by $\underline{A}^{\Theta I}$ and use of (30) then leads to

$$\dot{\lambda}^{\Phi} = \underline{A}^{\Phi I} \dot{a}^I \tag{33}$$

Thus once the total strain rate $\dot{\varepsilon}_{kl}$ is known, the plastic multipliers $\dot{\lambda}^{\Phi}$ can be determined. This implies that also the plastic strain rate and the rates of the internal variables are known via the evolution laws (15).

Insertion of (33) into Hooke's law (23) results in

$$\dot{\sigma}_{ij} = D^{ep}_{ijkl} \dot{\varepsilon}_{kl} \tag{34}$$

where the elasto-plastic stiffness tensor D^{ep}_{ijkl} is given by

$$D^{ep}_{ijkl} = D_{ijkl} - D_{ijst} \frac{\partial g^{\Phi}}{\partial \sigma_{st}} \underline{A}^{\Phi I} \frac{\partial f^I}{\partial \sigma_{mn}} D_{mnkl}. \tag{35}$$

It follows from (34) that the stress rate $\dot{\sigma}_{ij}$ is given uniquely by the total strain rate $\dot{\varepsilon}_{ij}$. It is of relevance, however, that when $F_{max} > G_{max}$, then the left inverse $\underline{A}^{\Phi I}$ is not known

*If A^{IJ} is positive definite then $Z^I A^{IJ} Z^J > 0$ for any $Z^I \neq 0$. We have $Z^I A^{IJ} Z^J = p_{ij} D_{ijkl} p_{kl}$ where $p_{ij} = Z^I \partial f^I / \partial \sigma_{ij}$; moreover, $p_{ij} D_{ijkl} p_{kl} > 0$ for any $p_{ij} \neq 0$. The requirement that $Z^I \neq 0$ implies $p_{ij} \neq 0$ is that $\partial f^I / \partial \sigma_{ij}$ are linearly independent; this requirement was already stated in (20).

uniquely; consequently the quantity D^{ep}_{ijkl} is not known uniquely. Nevertheless, noting requirement (28), the combined quantity $D^{ep}_{ijkl}\dot{\varepsilon}_{kl}$ is known uniquely.

5. CONSISTENT LOADING/UNLOADING CRITERIA

For smooth surfaces and referring to (24) and (33), the sign of the scalar quantity $\dot{a} = \partial f/\partial\sigma_{ij}D_{ijkl}\dot{\varepsilon}_{kl}$ determines whether we have plastic loading, neutral loading or elastic unloading. When corners exist, the situation becomes more complex.

Considering the special case of associated plasticity where $F_{max} = G_{max}$, Sewell (1973) and Simo *et al.* (1988) state that plastic loading, $\dot{\lambda}^I > 0$, occurs for some I-values if $\dot{a}^J > 0$ for some J-values. It follows from (33) that it is true that $\dot{a}^I = 0$ for all I-values implies $\dot{\lambda}^I = 0$ for all I values. However, Mandel (1965), Sewell (1973) and Simo *et al.* (1988) also note that even if $\dot{a}^{(I)} > 0$ for some specific I-value this does *not* necessarily imply that the corresponding plastic multiplier $\dot{\lambda}^{(I)} > 0$. In fact, other yield surfaces may be active. Therefore, the loading criterion $\dot{a}^I \geqslant 0$ is not a consistent loading criterion. For $F_{max} > G_{max}$, this viewpoint is even more evident.

To derive consistent loading/unloading criteria that hold for associated and non-associated plasticity where $F_{max} \geqslant G_{max}$, it turns out to be convenient to consider the cases $F_{max} = G_{max}$ and $F_{max} > G_{max}$ separately.

5.1 *The case where* $F_{max} = G_{max}$

When $F_{max} = G_{max}$, both associated and non-associated plasticity are characterized by the fact that to each specific potential function, there belongs one specific yield function. This means that for each number (I), we have

$$f^{(I)} \leqslant 0; \quad \dot{\lambda}^{(I)} \geqslant 0 \tag{36}$$

cf. (15) and (17). For associated plasticity, we also have

$$\dot{\lambda}^{(I)}f^{(I)} = 0 \quad \text{no summation} \tag{37}$$

cf. (19) and we shall assume that this relation also holds for non-associated plasticity.

It is trivial to identify the situation in which $f^I = 0$ holds for all yield functions, i.e., $\dot{f}^I \leqslant 0$. For that case, we shall now develop a search procedure for proper identification of loading or unloading. Taking the time derivative of (37), we obtain $\ddot{\lambda}^{(I)}f^{(I)} + \dot{\lambda}^{(I)}\dot{f}^{(I)} = 0$ and since $f^{(I)} = 0$, we find

$$\dot{\lambda}^{(I)}\dot{f}^{(I)} = 0 \quad \text{no summation.} \tag{38}$$

It appears that if a specific multiplier $\dot{\lambda}^{(I)} > 0$ then the consistency relation $\dot{f}^{(I)} = 0$ holds for the corresponding yield function. Likewise, $\dot{f}^{(I)} < 0$ implies $\dot{\lambda}^{(I)} = 0$.

Turning to matrix notation, i.e., $\dot{\lambda}^I = \dot{\boldsymbol{\lambda}}$, we may split the potential functions into active and passive functions according to

$$\dot{\boldsymbol{\lambda}} = \begin{bmatrix} \text{active potential functions} \\ \text{passive potential functions} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\lambda}}_a > 0 \\ \dot{\boldsymbol{\lambda}}_p = 0 \end{bmatrix} \tag{39}$$

where subscript a is the number of active potential functions and p is the number of passive potential functions, i.e., $a+p = G_{max} = F_{max}$. According to (38), $\dot{\lambda}^{(I)} > 0$ implies $\dot{f}^{(I)} = 0$, i.e., to each component of $\dot{\boldsymbol{\lambda}}_a$ we have a corresponding yield function for which $\dot{f}^{(I)} = 0$. This implies that it is also possible to split $\dot{f}^I = \dot{\mathbf{f}}$ according to

$$\dot{\mathbf{f}} = \begin{bmatrix} \dot{\mathbf{f}}_a = 0 \\ \dot{\mathbf{f}}_p \end{bmatrix}. \tag{40}$$

Following (38), it is true that $\dot{\lambda}^{(I)} > 0$ implies $\dot{f}^{(I)} = 0$. However, if $\dot{\lambda}^{(I)} = 0$ the situation is unspecified and we only know that $\dot{f}^{(I)} \leqslant 0$. Therefore, for $\dot{\mathbf{f}}_p$ we conclude that

$$\dot{\mathbf{f}}_p \leqslant 0. \tag{41}$$

Similar to (40) and (39), we make the following partitioning

$$\dot{\mathbf{a}} = \begin{bmatrix} \dot{\mathbf{a}}_a \\ \dot{\mathbf{a}}_p \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_{aa} & \mathbf{A}_{ap} \\ \mathbf{A}_{pa} & \mathbf{A}_{pp} \end{bmatrix} \tag{42}$$

where $\dot{a}^I$ was defined by (24). With (39)–(42), (24) may be written as

$$\dot{\mathbf{f}}_a = \dot{\mathbf{a}}_a - \mathbf{A}_{aa}\dot{\boldsymbol{\lambda}}_a = 0; \quad \dot{\boldsymbol{\lambda}}_a > 0 \tag{43}$$

$$\dot{\mathbf{f}}_p = \dot{\mathbf{a}}_p - \mathbf{A}_{pa}\dot{\boldsymbol{\lambda}}_a \leqslant 0. \tag{44}$$

The search procedure then amounts to identifying the active components "a" and thereby also of the passive ones $p = F_{max} - a$ such that (43) and (44) are fulfilled. Identification of the components "a" corresponds to plastic loading since $\dot{f}^{(I)} = 0$ and $\dot{\lambda}^{(I)} > 0$. Then, having solved (43), we check each component of $\dot{\mathbf{f}}_p$ in (44); if the specific component is zero, neutral loading occurs since $\dot{f}^{(I)} = 0$ and $\dot{\lambda}^{(I)} = 0$, and if the specific component is less than zero, we have elastic unloading since $\dot{f}^I < 0$ and $\dot{\lambda}^I = 0$.

5.2 *The case where $F_{max} > G_{max}$*

Turning to the case in which $F_{max} > G_{max}$, we have, according to (15) and (17), that

$$\dot{\lambda}^{\Phi} \geqslant 0; \quad f^I \leqslant 0. \tag{45}$$

This implies that

$$\text{if} f^I = 0 \quad \text{then} \ \dot{f}^I \leqslant 0. \tag{46}$$

When $F_{max} = G_{max}$, the identification of loading and unloading is facilitated by the fact that each potential function has a corresponding yield function and *vice versa*. This implies that if a specific multiplier $\dot{\lambda}^{(I)} > 0$ then $\dot{f}^{(I)} = 0$ holds for the corresponding yield function.

When $F_{max} > G_{max}$ the situation becomes more complex. First of all, it is necessary to identify the yield functions that belong to each potential function. Second, since more yield functions, in general, are related to one and the same potential function, from the fact that a certain multiplier $\dot{\lambda}^{(\Phi)} > 0$ we cannot conclude for which of the corresponding yield functions $\dot{f}^{(I)} = 0$ holds. However, as a generalization of (38) it seems natural to assume that

$$\begin{aligned} &\text{if } \dot{\lambda}^{(\Phi)} > 0 \text{ for some specific potential function then } \dot{f}^{(I)} = 0 \\ &\text{holds for at least one of the corresponding yield functions.} \end{aligned} \tag{47}$$

It appears that the search strategy for identification of loading and unloading becomes more involved than that provided by (43) and (44).

It is trivial to identify the situation in which $f^I = 0$ holds for all yield functions, i.e., $\dot{f}^I \leqslant 0$, as will be assumed in the following discussion.

Turning to matrix notation, i.e., $\dot{f}^I = \dot{\mathbf{f}}$, we may split the yield functions into active and passive ones according to

$$\dot{\mathbf{f}} = \begin{bmatrix} \text{active yield functions} \\ \text{passive yield functions} \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{f}}_a = 0 \\ \dot{\mathbf{f}}_p < 0 \end{bmatrix} \tag{48}$$

where a denotes the number of active yield functions and p the number of passive yield functions. It follows that $a + p = F_{max}$. With $\dot{a}^I$ defined by (24), a similar decomposition can be performed on $\dot{\mathbf{a}}$:

$$\dot{\mathbf{a}} = \begin{bmatrix} \dot{\mathbf{a}}_a \\ \dot{\mathbf{a}}_p \end{bmatrix} \tag{49}$$

the following decomposition is made of $\dot{\lambda}^\Phi = \dot{\boldsymbol{\lambda}}$:

$$\dot{\boldsymbol{\lambda}} = \begin{bmatrix} \text{active potential functions} \\ \text{passive potential functions} \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\lambda}}_{\underline{a}} > 0 \\ \dot{\boldsymbol{\lambda}}_{\underline{p}} = 0 \end{bmatrix} \tag{50}$$

where subscript $\underline{a}$ denotes the number of active potential functions and $\underline{p}$ is the number of passive potential functions, i.e. $\underline{a} + \underline{p} = G_{max}$. Note that, in general, we have $a \geqslant \underline{a}$. With evident notation, the matrix $A^{I\Phi} = \mathbf{A}$ is partitioned as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{a\underline{a}} & \mathbf{A}_{a\underline{p}} \\ \mathbf{A}_{p\underline{a}} & \mathbf{A}_{p\underline{p}} \end{bmatrix}. \tag{51}$$

With (48)–(51), (24) may be written as

$$\dot{\mathbf{f}}_a = \dot{\mathbf{a}}_a - \mathbf{A}_{a\underline{a}} \dot{\boldsymbol{\lambda}}_{\underline{a}} = 0; \quad \dot{\boldsymbol{\lambda}}_{\underline{a}} > 0 \tag{52}$$

$$\dot{\mathbf{f}}_p = \dot{\mathbf{a}}_p - \mathbf{A}_{p\underline{a}} \dot{\boldsymbol{\lambda}}_{\underline{a}} < 0. \tag{53}$$

Noting that $p = F_{max} - a$, the search procedure now amounts to identifying both of the components a and $\underline{a}$ such that (52) and (53) are fulfilled. Moreover, (47) also needs to be satisfied. In general, $a \geqslant \underline{a}$, i.e., $\mathbf{A}_{a\underline{a}}$ in (52) may have more rows than columns; if this is the case, a solution of (52) requires that $\dot{\mathbf{a}}_a$ be written as linear combinations of the columns of $\mathbf{A}_{a\underline{a}}$, cf. the discussion related to (28).

In order to characterize the different loading/unloading conditions, we first introduce the following notation:

$$\begin{bmatrix} \text{To a specific potential function } g^{(\Phi)}\text{, i.e., a specific multiplier } \dot{\lambda}^{(\Phi)}, \\ \qquad\qquad \text{the corresponding yield functions are denoted by } f^I_{(\Phi)} \end{bmatrix} \tag{54}$$

We then obtain

$$\text{if} \quad \dot{\lambda}^{(\Phi)} > 0 \quad \text{then} \begin{cases} \text{if all } \dot{f}^I_{(\Phi)} = 0 \\ \qquad \text{then fully plastic loading} \\ \qquad \text{with all yield functions being active} \\ \text{if some } \dot{f}^I_{(\Phi)} = 0 \text{ and some } \dot{f}^I_{(\Phi)} < 0 \\ \qquad \text{then partly plastic loading} \\ \qquad \text{with some yield functions being passive.} \end{cases} \tag{55}$$

(We observe that, according to (47), we cannot have $\dot{\lambda}^{(\Phi)} > 0$ and all $\dot{f}^I_{(\Phi)} < 0$.)

$$\text{if}\quad \dot{\lambda}^{(\Phi)}=0\quad\text{then}\quad \begin{cases} \text{if all } \dot{f}^{I}_{(\Phi)}=0 \\ \quad \text{then fully neutral loading} \\ \quad \text{with all yield functions being active} \\ \text{if some } \dot{f}^{I}_{(\Phi)}=0 \text{ and some } \dot{f}^{I}_{(\Phi)}<0 \\ \quad \text{then partly neutral loading} \\ \quad \text{with some yield functions being passive} \\ \text{if all } \dot{f}^{I}_{(\Phi)}<0 \\ \quad \text{then elastic unloading} \\ \quad \text{with all yield functions being passive.} \end{cases} \tag{56}$$

6. EVALUATION OF RESPONSE

The properties of the matrix $A^{I\Phi}$ are given by (27), (31) and (32). However, it also turns out to be extremely important to evaluate the properties of the plastic moduli matrix $H^{I\Phi}$ defined by (22). As a preliminary to that, we shall study the response as prescribed by (34).

Let us consider the possibility that $\dot{\varepsilon}_{ij}\neq 0$ may imply $\dot{\sigma}_{ij}=0$. For evident reasons the corresponding state is termed a limit point

$$0=\dot{\sigma}_{ij}=D^{ep}_{ijkl}\dot{\varepsilon}_{kl}=0\quad\text{limit point.} \tag{57}$$

Since $\dot{\sigma}_{ij}=0$, Hooke's law implies $\dot{\varepsilon}_{ij}=\dot{\varepsilon}^{p}_{ij}$. Use of (15) then shows that the solution must be of the form

$$\dot{\varepsilon}_{ij}=\dot{\lambda}^{\Phi}\frac{\partial g^{\Phi}}{\partial\sigma_{ij}}. \tag{58}$$

Using the limit condition $\dot{\sigma}_{ij}=0$ in the consistency relation $\dot{f}^{I}=0$, we obtain from (21)

$$H^{I\Phi}\dot{\lambda}^{\Phi}=0. \tag{59}$$

Since $F_{max}\geqslant G_{max}$, the rank of $H^{I\Phi}\leqslant G_{max}$. If the rank of $H^{I\Phi}=G_{max}$, then (59) only possesses the trivial solution $\dot{\lambda}^{\Phi}=0$, which is of no interest. We therefore conclude that

$$\text{existence of limit point}\Leftrightarrow\text{rank of } H^{I\Phi}<G_{max};\quad \dot{\lambda}^{\Phi}\geqslant 0. \tag{60}$$

Relevant solutions of (59) are clearly subject to the conditions $\dot{\lambda}^{\Phi}\geqslant 0$.

Next let us consider the special case of perfect plasticity. In that case, no conjugated forces A_{α} exist in the yield functions and (22) provides

$$H^{I\Phi}=0\Leftrightarrow\text{perfect plasticity} \tag{61}$$

i.e., all components of $H^{I\Phi}$ are zero. This means that (59) has solutions for arbitrary $\dot{\lambda}^{\Phi}$-values, i.e. the total strain rate given by (58) can take infinitely many values and still have $\dot{\varepsilon}_{ij}\neq 0$ which implies $\dot{\sigma}_{ij}=0$. The only restriction on $\dot{\varepsilon}_{ij}$ is that we must have plastic loading, i.e. $\dot{\lambda}^{\Phi}\geqslant 0$. Therefore when corners exist, there are more values of $\dot{\varepsilon}_{ij}\neq 0$ which imply $\dot{\sigma}_{ij}=0$, as compared with smooth surfaces where the direction of $\dot{\varepsilon}_{ij}$ is given uniquely by $\partial g/\partial\sigma_{ij}$.

We may also note another difference between corner plasticity and conventional plasticity with no corners. For conventional plasticity, perfect plasticity ($H=0$) and a limit point are identical statements. For corner plasticity, a glance at (60) and (61) reveals a fundamental difference, since perfect plasticity implies the existence of a limit point whereas the reverse is not true.

7. STRESS DRIVEN FORMAT

Considering (21), the consistency relations $\dot{f}^I = 0$ provide

$$H^{I\Phi}\dot{\lambda}^{\Phi} = \frac{\partial f^I}{\partial \sigma_{ij}}\dot{\sigma}_{ij}. \tag{62}$$

In order to be able to derive a stress driven format, it is necessary that this inhomogeneous equation system allow for a unique $\dot{\lambda}^{\Phi}$-solution. Since $H^{I\Phi}$ has the dimension $(F_{max} \times G_{max})$, the augmented matrix **T** has the dimension $(F_{max} \times (G_{max}+1))$. A unique solution of (62), where G_{max} is the number of unknowns requires that Rank (**A**) = Rank (**T**) = G_{max}, i.e.,

$$\text{stress driven format requires that rank of } H^{I\Phi} = G_{max} \tag{63}$$

and

$$\begin{array}{l}\text{stress driven format requires that } \partial f^I/\partial\sigma_{ij}\dot{\sigma}_{ij} \text{ can be}\\ \text{expressed by linear combinations of the columns of } H^{I\Phi}.\end{array} \tag{64}$$

If $F_{max} = G_{max}$ and (63) is satisfied, then (64) is fulfilled trivially.

Assuming that (63) is fulfilled, $H^{I\Phi}$ possesses a left inverse $\underline{H}^{\Theta I}$ defined by

$$\underline{H}^{\Theta I}H^{I\Phi} = \delta^{\Theta\Phi}. \tag{65}$$

Multiplication of (62) by $\underline{H}^{\Theta I}$ then leads to

$$\dot{\lambda}^{\Phi} = \underline{H}^{\Phi I}\frac{\partial f^I}{\partial \sigma_{ij}}\dot{\sigma}_{ij}. \tag{66}$$

Hooke's law (12) may also be written as

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij} + \dot{\varepsilon}^p_{ij} = C_{ijkl}\dot{\sigma}_{kl} + \dot{\varepsilon}^p_{ij} \tag{67}$$

where C_{ijkl} is the elastic flexibility tensor. With (15) and (66), we then obtain

$$\dot{\varepsilon}_{ij} = C^{ep}_{ijkl}\dot{\sigma}_{kl} \tag{68}$$

where the elasto-plastic flexibility tensor C^{ep}_{ijkl} is given by

$$C^{ep}_{ijkl} = C_{ijkl} + \frac{\partial g^{\Phi}}{\partial \sigma_{ij}}\underline{H}^{\Phi I}\frac{\partial f^I}{\partial \sigma_{kl}}. \tag{69}$$

With regard to a unique expression for C^{ep}_{ijkl}, we refer to the similar discussion following (35).

It is of interest to compare the requirements for obtaining a strain driven format and a stress driven format. A stress driven format is not possible at a limit point or for perfect plasticity, cf. (60), (61) with (63). More generally, comparing (27) and (63) and observing that the term $\partial f^I/\partial\sigma_{ij}D_{ijkl}\partial g^{\Phi}/\partial\sigma_{kl}$ present in $A^{I\Phi}$ is positive definite for associated plasticity, it is evident that the requirements for existence of a strain driven format are much less severe than those related to a stress driven format.

8. PROPERTIES OF THE PLASTIC MODULI MATRIX

Some preliminary consequences of the properties of the plastic moduli matrix $H^{I\Phi}$ were given by (60), (61) and (63). We are now in a position to scrutinize the properties of the matrix $H^{I\Phi}$ and to discuss various consequences.

It is of interest that these properties have only been discussed to a minor degree in the literature. For associated plasticity, which implies $F_{max} = G_{max}$ and that H^{IJ} is symmetric, Mandel (1965) proved that if H^{IJ} is positive definite then uniqueness follows and Hill (1966) showed that if H^{IJ} is positive definite, then both a stress and a strain driven format exist; these consequences also follow from our previous discussion. For associated plasticity, Sewell (1973, 1974) argued that H^{IJ} should be positive definite whereas Simo *et al.* (1988) confined their comments to the structure of the expression for H^{IJ}. We shall return to these comments later on. Here, we shall extend the discussion to general non-associated plasticity where $F_{max} \geqslant G_{max}$ and we shall propose concise definitions of hardening, perfect and softening plasticity.

Limit points and perfect plasticity have already been defined by (60) and (61), respectively. Taking the strain driven format as the most fundamental and general formulation, we start out with the postulate that

$$\text{hardening plasticity allows for a stress driven format} \tag{70}$$

This postulate seems quite natural, but it excludes the case of so-called Taylor hardening and we shall return to this degenerated case in the application sections.

In order to pursue the implications of (70), we first investigate the conditions when the elasto-plastic response approaches the linear elastic response. In that limit, elastic response is obtained if all the components of $A^{\Theta I}$ approach zero, cf. (35). Referring to (30), $A^{\Theta I} \to 0$ implies that some of the components of $A^{I\Phi}$ become infinitely large, i.e., $A^{I\Phi} \to \pm\infty$. Since the quantities $\partial f^I/\partial\sigma_{ij} D_{ijkl} \partial g^\Phi/\partial\sigma_{kl}$ are finite quantities, (25) shows that the only possibility for $A^{I\Phi} \to \pm\infty$ to be achieved is that the components of $H^{I\Phi} \to \pm\infty$. However, according to (31) and (32), the part $P^{\Theta\Phi}$ of $A^{I\Phi}$ possesses positive eigenvalues, i.e., when elastic response is approached, we must require that $H^{I\Phi} \to \infty$.

In analogy with (31), we may write $H^{I\Phi}$ according to

$$H^{I\Phi} = \begin{bmatrix} R^{\Theta\Phi} \\ S^{Y\Phi} \end{bmatrix} \tag{71}$$

where $R^{\Theta\Phi}$ is of dimension $(G_{max} \times G_{max})$ and $S^{Y\Phi}$ is of dimension $((F_{max} - G_{max}) \times G_{max})$. According to (63) and (70), it is always possible to choose the numbering of the yield functions such that $R^{\Theta\Phi}$ for hardening has the rank G_{max}. In the limit of linear elasticity, we found above that $H^{I\Phi} \to \infty$. Moreover, we expect hardening plasticity to exist in the process where linear elasticity is approached. This suggests the following definition

$$\text{hardening plasticity} \Leftrightarrow R^{\Theta\Phi} \text{ has only positive eigenvalues.} \tag{72}$$

With reference to (60), the existence of a limit point may be reformulated as

$$\text{limit point} \Leftrightarrow R^{\Theta\Phi} \text{ has at least one zero eigenvalue.} \tag{73}$$

We are then left with the following conclusion

$$\text{softening plasticity} \Leftrightarrow R^{\Theta\Phi} \text{ has at least one negative eigenvalue and no eigenvalues are zero.} \tag{74}$$

It is recalled from (61) that perfect plasticity exists if all the components of $H^{I\Phi}$ are zero.

Finally, if we have associated plasticity, i.e., $F_{max} = G_{max}$ and $f^I = g^I$, then (22) and (25) show that

$$\text{associated plasticity} \Leftrightarrow H^{IJ} \text{ and } A^{IJ} \text{ are symmetric matrices.} \tag{75}$$

We shall later investigate the uniqueness properties and this analysis will turn out to support the conclusions above. However, as a first illustration of these properties, it is of considerable interest to compare (66) with the corresponding formulation (4) of Koiter. It appears that the two formulations coincide for associated plasticity where $F_{max} = G_{max}$ if and only if $\underline{H}^{IJ}$ is a diagonal matrix. In that case $\dot{\lambda}^{(I)}$ is influenced only by the corresponding yield function $f^{(I)}$. Moreover, in accordance with (4), if $\underline{H}^{IJ}$ is diagonal then all components of $\underline{H}^{IJ}$ are positive for hardening plasticity; otherwise (72) would be violated. This means the loading condition $\dot{\lambda}^{(I)} \geqslant 0$ for yield surface number I is equivalent to the requirement that $\partial f^{(I)}/\partial\sigma_{ij}\dot{\sigma}_{ij} \geqslant 0$.

As the next illustration, consider hardening associated plasticity where H^{IJ} is symmetric and positive definite. Multiplication of the consistency relation $\dot{f}^I = 0$ by $\dot{\lambda}^I$ gives with (21) that $\dot{\lambda}^I\partial f^I/\partial\sigma_{ij}\dot{\sigma}_{ij} = \dot{\lambda}^I H^{IJ}\dot{\lambda}^J$. Use of the flow rule then asserts that $\dot{\varepsilon}^p_{ij}\dot{\sigma}_{ij} > 0$ holds for hardening associated plasticity and this result is in agreement with the postulate of Drucker (1951).

Let us finally return to the comments of Simo *et al.* (1988) about the structure of $H^{I\Phi}$ given by (22). Simo *et al.* considered associated plasticity and arrived at the same format for H^{IJ} as (22), except that they did not adopt a formulation based upon thermodynamics and they used the notation $\mathbf{D}$ for $\partial^2\psi^p/\partial\kappa_\alpha\partial\kappa_\beta$. In accord with the present formulation, Simo *et al.* (1988) observed that $\mathbf{D}$ is symmetric, but also that $\mathbf{D}$ can be assumed to be constant and positive definite. The present formulation does not provide support for these latter two properties, as they would exclude the existence of limit points and of softening. However, for hardening associated plasticity, $\partial^2\psi^p/\partial\kappa_\alpha\partial\kappa_\beta$ is certainly positive definite as apparent from (22), (20) and (72).

9. UNIQUENESS

It is also of interest to investigate the uniqueness problem, i.e., to investigate whether two possible solutions exist for a given boundary value problem. If two different solutions exist they must satisfy the requirement

$$\int_V I\,\mathrm{d}V = 0 \quad \text{where } I = [\dot{\varepsilon}_{ij}][\dot{\sigma}_{ij}] \tag{76}$$

and $[\dot{\varepsilon}_{ij}]$ and $[\dot{\sigma}_{ij}]$ denote the difference between the two solutions, i.e., $[\dot{\varepsilon}_{ij}] = \dot{\varepsilon}^{(2)}_{ij} - \dot{\varepsilon}^{(1)}_{ij}$ and $[\dot{\sigma}_{ij}] = \dot{\sigma}^{(2)}_{ij} - \dot{\sigma}^{(1)}_{ij}$. Koiter (1953, 1960) proved that formulation (4) leads to uniqueness and for associated plasticity, Mandel (1965) showed that if H^{IJ} is positive definite, then uniqueness follows. Here we shall derive an explicit criterion for uniqueness that holds for general non-associated plasticity.

Assuming that the current state is the same for the two solutions we obtain with (34) that

$$I = [\dot{\varepsilon}_{ij}]D^{ep}_{ijkl}[\dot{\varepsilon}_{kl}]. \tag{77}$$

In this form only the symmetric part $D^{ep,sym}_{ijkl}$ of D^{ep}_{ijkl} contributes. Therefore, if the symmetric part of D^{ep}_{ijkl} is positive definite, $I > 0$ for any $[\dot{\varepsilon}_{ij}] \neq 0$, and requirement (76) implies that we must have $[\dot{\varepsilon}_{ij}] = 0$, i.e., uniqueness in strains and thereby also uniqueness in stresses. As a consequence, uniqueness is lost as soon as one of the eigenvalues of $D^{ep,sym}_{ijkl}$ becomes zero.

Let us therefore investigate the eigenvalue problem

$$D^{cp,sym}_{ijkl}[\dot{\varepsilon}_{kl}] = \mu D_{ijkl}[\dot{\varepsilon}_{kl}]. \tag{78}$$

In reality, we need to investigate the eigenvalue problem $D^{cp,sym}_{ijkl}[\dot{\varepsilon}_{kl}] = \alpha[\dot{\varepsilon}_{ij}]$. However, since D_{ijkl} is positive definite $\alpha = 0$ implies $\mu = 0$ and *vice versa*. Introducing the notation

$$a^I = \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl}[\dot{\varepsilon}_{kl}]\,; \quad b^\Phi = \frac{\partial g^\Phi}{\partial \sigma_{ij}} D_{ijkl}[\dot{\varepsilon}_{kl}] \tag{79}$$

use of (35) in (78) yields

$$(1-\mu)D_{ijkl}[\dot{\varepsilon}_{kl}] - \frac{1}{2}\left(D_{ijst}\frac{\partial g^\Phi}{\partial \sigma_{st}}\underline{A}^{\Phi I}a^I + b^\Phi \underline{A}^{\Phi I}\frac{\partial f^I}{\partial \sigma_{mn}}D_{mnij}\right) = 0 \tag{80}$$

where we utilize the symmetry of D_{ijkl}, i.e., $D_{ijkl} = D_{klij}$. Let us also introduce the notation

$$F^{IJ} = \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijst}\frac{\partial f^J}{\partial \sigma_{st}}; \quad G^{\Theta\Phi} = \frac{\partial g^\Theta}{\partial \sigma_{ij}} D_{ijst}\frac{\partial g^\Phi}{\partial \sigma_{st}}; \quad M^{I\Phi} = \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijst}\frac{\partial g^\Phi}{\partial \sigma_{st}}. \tag{81a,b,c}$$

Since D_{ijkl} is symmetric and positive definite, it follows that both F^{IJ} and $G^{\Theta\Phi}$ are symmetric and positive definite matrices. (Here we take advantage of (20), cf. the discussion leading to expression (32).) They therefore possess inverse matrices defined by

$$\underline{F}^{KI}F^{IJ} = \delta^{KJ}; \quad \underline{G}^{\Psi\Theta}G^{\Theta\Phi} = \delta^{\Psi\Phi} \tag{82}$$

and no difference exists between left and right inverses. Multiplication of (80) by $\partial g^\Psi/\partial \sigma_{ij}$ then yields with (79) as well as (81b) and (81c) that

$$G^{\Psi\Phi}\underline{A}^{\Phi I}a^I = [2(1-\mu)\delta^{\Psi\Phi} - \underline{A}^{\Phi S}M^{S\Psi}]b^\Phi.$$

From (26) we have $\underline{A}^{\Phi I}a^I = [\dot{\lambda}]^\Phi$, i.e.

$$G^{\Psi\Phi}[\dot{\lambda}]^\Phi = [2(1-\mu)\delta^{\Psi\Phi} - \underline{A}^{\Phi S}M^{S\Psi}]b^\Phi. \tag{83}$$

Likewise, multiplication of (80) by $\partial f^K/\partial \sigma_{ij}$ and use of (79) along with (81a) and (81c) result in

$$[2(1-\mu)\delta^{KI} - M^{K\Omega}\underline{A}^{\Omega I}]a^I = b^\Omega \underline{A}^{\Omega I}F^{IK}. \tag{84}$$

Multiplication by $\underline{F}^{KJ}$ yields

$$[2(1-\mu)\underline{F}^{IJ} - M^{K\Omega}\underline{A}^{\Omega I}\underline{F}^{KJ}]a^I = b^\Omega \underline{A}^{\Omega J}.$$

Finally, since $a^I = A^{I\Theta}[\dot{\lambda}]^\Theta$ multiplication by $A^{J\Phi}$ provides

$$[2(1-\mu)\underline{F}^{IJ}A^{I\Theta}A^{J\Phi} - M^{K\Theta}\underline{F}^{KJ}A^{J\Phi}][\dot{\lambda}]^\Theta = b^\Phi. \tag{85}$$

Use of (85) in (83) then yields

$$B^{\Psi\Theta}[\dot{\lambda}]^\Theta = 0 \tag{86}$$

where the matrix $B^{\Psi\Theta}$ is defined by

$$B^{\Psi\Theta} = [4(1-\mu)^2 \underline{F}^{IJ} A^{J\Psi} A^{I\Theta} - 2(1-\mu) \underline{F}^{IJ} A^{J\Phi} \underline{A}^{\Phi S} M^{S\Psi} A^{I\Theta} - G^{\Psi\Theta} \\ -2(1-\mu) M^{K\Theta} \underline{F}^{KJ} A^{J\Psi} + M^{K\Theta} \underline{F}^{KJ} A^{J\Phi} \underline{A}^{\Phi S} M^{S\Psi}]. \tag{87}$$

It is evident that the trivial solution $[\dot{\lambda}]^{\Theta} = 0$ satisfies (86). However, if $[\dot{\lambda}]^{\Theta} = 0$ then (85) implies $b^{\Phi} = 0$ and (26) implies $a^I = 0$. Moreover, (80) is then fulfilled for the eigenvalue

$$\mu = 1. \tag{88}$$

Returning to the eigenvalue problem (78), uniqueness in the strain rate is lost if $\mu = 0$. Loss of uniqueness is therefore obtained for

$$V^{\Psi\Theta}[\dot{\lambda}]^{\Theta} = 0; \quad [\dot{\lambda}]^{\Theta} \geqslant 0 \tag{89}$$

where $V^{\Psi\Theta} = B^{\Psi\Theta}(\mu = 0)$, i.e.

$$V^{\Psi\Theta} = 4\underline{F}^{IJ} A^{J\Psi} A^{I\Theta} - 2\underline{F}^{IJ} A^{J\Phi} \underline{A}^{\Phi S} M^{S\Psi} A^{I\Theta} - G^{\Psi\Theta} \\ -2M^{K\Theta} \underline{F}^{KJ} A^{J\Psi} + M^{K\Theta} \underline{F}^{KJ} A^{J\Phi} \underline{A}^{\Phi S} M^{S\Psi}. \tag{90}$$

It is concluded that

$$\text{loss of uniqueness requires } \det V^{\Psi\Theta} = 0. \tag{91}$$

With this general result, let us investigate some special situations.

9.1 *No corners*

For smooth yield and potential functions, i.e., no corners, we have $F_{max} = G_{max} = 1$. With (91), loss of uniqueness then reduces to $V = 0$, i.e.

$$4\underline{F}AA - 2\underline{F}A\underline{A}MA - G - 2M\underline{F}A + M\underline{F}A\underline{A}M = 0$$

with evident notation. Since, for instance $\underline{F} = F^{-1}$, we obtain

$$4A^2 - 4MA - GF + M^2 = 0.$$

As $A = H + M$, cf. (25) and (81c), we find that loss of uniqueness occurs when the plastic modulus H is given by

$$H = \tfrac{1}{2}(-M \pm \sqrt{GF}). \tag{92}$$

In order to evaluate this expression in more detail, we define the tensor T_{ijkl} by

$$T_{ijmn} T_{mnkl} = D_{ijkl}. \tag{93}$$

Since D_{ijkl} is positive definite, this factorization is always possible. Next we define the following quantities:

$$p_{kl} = \frac{\partial f}{\partial \sigma_{ij}} T_{ijkl}; \quad q_{kl} = \frac{\partial g}{\partial \sigma_{ij}} T_{ijkl}. \tag{94}$$

Referring to (81), it then follows that

$$F = p_{ij}p_{ij}; \quad G = q_{ij}q_{ij}; \quad M = p_{ij}q_{ij}. \tag{95}$$

Moreover, from (94) we define the following unit tensors

$$m_{ij} = \frac{p_{ij}}{\|p\|}; \quad n_{ij} = \frac{q_{ij}}{\|q\|}; \quad \|p\| = (p_{ij}p_{ij})^{1/2}; \quad \|q\| = (q_{ij}q_{ij})^{1/2}. \tag{96}$$

It follows that $m_{ij}m_{ij} = 1$ and $n_{ij}n_{ij} = 1$. The angle θ is defined by

$$m_{ij}n_{ij} = \cos\theta. \tag{97}$$

It is observed that $\theta = 0$ for associated plasticity. With (95)–(97), we have

$$F = \|p\|^2; \quad G = \|q\|^2; \quad M = \|p\|\,\|q\|\cos\theta. \tag{98}$$

In practice, we are interested in the earliest situation where uniqueness is lost. Therefore, using the upper sign in (92), as well as (98), we find

$$H = \tfrac{1}{2}\|p\|\,\|q\|\,(1-\cos\theta). \tag{99}$$

With this result, we have retrieved the classical result by Maier and Hueckel (1979). We may note that for associated plasticity, i.e., $\theta = 0$, (99) implies that uniqueness is lost when $H = 0$. For non-associated plasticity, however, uniqueness is lost even during hardening where $H > 0$.

9.2 *Associated plasticity*

Consider next associated corner plasticity. In that case we have $F_{max} = G_{max}$ as well as $f^I = g^I$, i.e.

$$F^{IJ} = G^{IJ} = M^{IJ} \tag{100}$$

where these matrices are symmetric and positive definite, cf. (81). Moreover, A^{IJ} now becomes a symmetric matrix and we do not have to differentiate between the left and right inverses. For this case, (89) reduces to

$$(\underline{F}^{IJ}A^{JL}A^{IM} - A^{ML})[\dot{\lambda}]^M = 0.$$

Multiplication by $\underline{A}^{LK}$ yields

$$(\underline{F}^{IK}A^{IM} - \delta^{MK})[\dot{\lambda}]^M = 0.$$

Finally, multiplication by F^{JK} and since $A^{JM} = H^{JM} + F^{JM}$, we obtain

$$H^{JM}[\dot{\lambda}]^M = 0. \tag{101}$$

i.e., $\det H^{JM} = 0$ is required for loss of uniqueness. Referring to (72), hardening associated corner theory will always give rise to uniqueness. The only possibility for loss of uniqueness for associated plasticity is when a limit point is encountered, cf. (73).

If I given by (76) is written as $I = [\dot{\sigma}_{ij}]C^{ep}_{ijkl}[\dot{\sigma}_{kl}]$ and we employ (69) and (72), uniqueness of hardening associated plasticity may, in fact, be demonstrated directly, as also shown by Koiter (1953, 1960) and Mandel (1965).

These results imply that hardening associated plasticity is characterized by uniqueness and that a stress driven format is possible; moreover Drucker's criterion $\dot{\varepsilon}^p_{ij}\dot{\sigma}_{ij} > 0$ holds. This supports the previously stated definition of hardening plasticity.

9.3 *Two yield functions—one potential function*

Let us next consider the important situation often encountered in practice, namely two yield functions and one potential function. In this case, condition (91) for non-uniqueness reduces to

$$V^{11} = 0. \tag{102}$$

This condition applies even for several yield functions and one potential function.

9.4 *Two yield functions—two potential functions*

This case also is important in practice and we have $F_{max} = G_{max} = 2$. Consider the following eigenvalue problem

$$V^{\Psi\Theta}c^{\Theta} = \mu c^{\Psi}. \tag{103}$$

Referring to (89), we are interested in the situation where $\mu = 0$ which leads to the following more explicit condition for non-uniqueness

$$(V^{\Psi\Psi})^2 = V^{\Theta\Psi}V^{\Psi\Theta}. \tag{104}$$

10. TRESCA YIELD FUNCTION—TRESCA POTENTIAL FUNCTION

In order to illustrate some of our findings, let us adopt associated plasticity of a Tresca material, i.e., $F_{max} = G_{max} = 2$. With σ_1, σ_2 and σ_3 being the principal stresses, the two yield functions are given by

$$\begin{aligned} f^1 &= \sigma_1 - \sigma_3 + k_{1\beta}A_\beta - \sigma_{yo} = 0 \\ f^2 &= \sigma_2 - \sigma_3 + k_{2\beta}A_\beta - \sigma_{yo} = 0 \end{aligned} \tag{105}$$

and corner loading occurs when

$$\sigma_1 = \sigma_2 + (k_{2\beta} - k_{1\beta})A_\beta. \tag{106}$$

In (105), $k_{1\beta}$ and $k_{2\beta}$ are constant quantities and σ_{yo} denotes the initial yield stress. When plasticity is just initiated, the conjugated forces A_β are zero and we then have $\sigma_1 = \sigma_2 > \sigma_3$ where tension is considered positive.

Helmholtz's free energy is taken in the form

$$\psi = \tfrac{1}{2}(\varepsilon_{ij} - \varepsilon_{ij}^p)D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^p) + \psi^p(\kappa_\alpha) \tag{107}$$

where D_{ijkl} is the constant elasticity tensor, cf. (7). From (9) and (11) it follows that

$$\sigma_{ij} = D_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^p); \quad A_\alpha = -\frac{\partial \psi^p}{\partial \kappa_\alpha}. \tag{108}$$

The format (105) implies that each yield surface hardens in an isotropic manner. Since two yield functions exist, it is natural to assume the existence of two conjugated forces A_1 and A_2. Expression (108) then shows the existence of two internal variables κ_1 and κ_2, i.e.

$$A_\alpha \in A_1 \text{ and } A_2; \quad \kappa_\alpha \in \kappa_1 \text{ and } \kappa_2. \tag{109}$$

From (105) it follows that

$$\frac{\partial f^I}{\partial A_\alpha} = k_{I\alpha} \tag{110}$$

and (15) then yields

$$\dot{\kappa}_\alpha = \dot{\lambda}^I k_{I\alpha}. \tag{111}$$

For convenience we introduce the notation

$$\psi_{\alpha\beta} = \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \, \partial \kappa_\beta} \tag{112}$$

where $\psi_{\alpha\beta}$ is symmetric. Use of (111) and (112) in (22) then provides the following form of the matrix of plastic moduli

$$H^{IJ} = k_{I\alpha}\psi_{\alpha\beta}k_{\beta J}. \tag{113}$$

A natural choice of hardening seems to be given by the assumption that the two yield surfaces expand by the same amount irrespective of whether we have loading only along one of the smooth yield surfaces or corner loading, cf. Fig. 2. The fact that the two yield surfaces exhibit the same amount of isotropic hardening can be modelled by the following form of $k_{I\alpha}$

$$k_{I\alpha} = \begin{bmatrix} k & k \\ k & k \end{bmatrix}. \tag{114}$$

Insertion into (113) proides

$$H^{IJ} = k^2(\psi_{11} + 2\psi_{12} + \psi_{22})\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \text{Taylor hardening.} \tag{115}$$

It appears that det $H^{IJ} = 0$, i.e., rank$(H^{IJ}) < G_{max} = 2$ and two important consequences follow. Referring to (63), a stress driven formulation is not possible, and even if a strain driven format is accepted, uniqueness is lost for any values of k and $\psi_{\alpha\beta}$, cf. (101). The

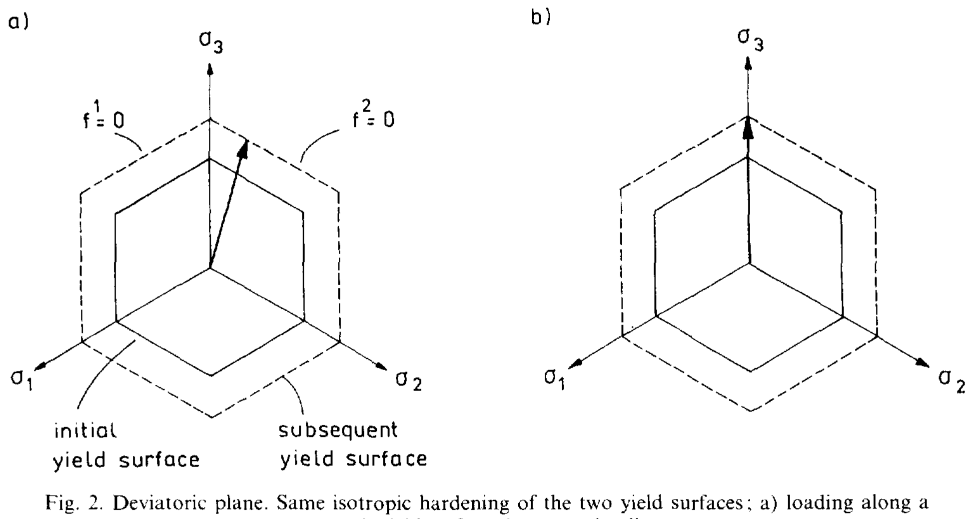

Fig. 2. Deviatoric plane. Same isotropic hardening of the two yield surfaces; a) loading along a smooth yield surface; b) corner loading.

hardening indicated by (115) is called Taylor hardening (1938) and even though it corresponds to a type of isotropic hardening that seems quite natural, cf. Fig. 2, this model is not appropriate.

Another choice of isotropic hardening is that where hardening of one yield surface does not influence the other yield surfaces. This type of independent isotropic hardening is illustrated in Fig. 3. Referring to (105) this hardening can be modelled by assuming that

$$k_{I\alpha} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \quad \text{i.e.,} \ k_{I\alpha} = k\delta_{I\alpha}. \tag{116}$$

It is of interest that even if radial loading initially occurs along a smooth yield surface, eventually it will result in corner loading. With (113) and (116), we obtain

$$H^{IJ} = k^2\psi_{IJ}. \tag{117}$$

It seems natural to assume that $\psi_{11} = \psi_{22}$, i.e.,

$$H^{IJ} = k^2\begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{12} & \psi_{11} \end{bmatrix}. \tag{118}$$

Again, if we choose $\psi_{11} = \psi_{12}$ then a stress driven format is not possible; therefore $\psi_{12} \neq \psi_{11}$ is assumed. If $\psi_{12} = \psi_{11}/2$, then, following Sewell (1973), this may be termed Budiansky-Wu hardening (1962). If it is assumed that $\psi_{12} = 0$, then

$$H^{IJ} = k^2\psi_{11}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \text{Koiter hardening} \tag{119}$$

i.e., Koiter hardening, cf. (4). As long as $k^2\psi_{11} \neq 0$, a stress driven format is possible. If $k^2\psi_{11} > 0$ then hardening plasticity occurs and uniqueness is ensured, cf. (72) and (101); if $k^2\psi_{11} < 0$, softening occurs according to (74). Finally, from (73) it appears that existence of a limit point requires $k^2\psi_{11} = 0$ which even corresponds to perfect plasticity, cf. (61).

In order to evaluate the situation, in which fully plastic loading occurs for both yield surfaces, we note from (105) that

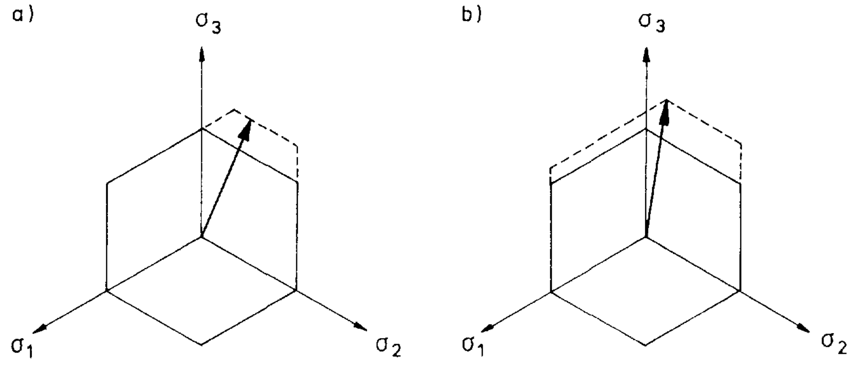

Fig. 3. Deviatoric plane. Independent isotropic hardening of each yield function: a) loading along a smooth yield surface; b) corner loading.

$$\frac{\partial f^1}{\partial \sigma_{ij}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad \frac{\partial f^2}{\partial \sigma_{ij}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{120}$$

Considering isotropic elasticity, we have

$$D_{ijkl} = 2G\left[\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+\frac{v}{1-2v}\delta_{ij}\delta_{kl}\right] \tag{121}$$

where G = shear modulus and v = Poisson's ratio. With (25) and (119), it then follows that

$$A^{IJ} = \begin{bmatrix} k^2\psi_{11}+4G & 2G \\ 2G & k^2\psi_{11}+4G \end{bmatrix}; \quad \frac{\partial f^I}{\partial \sigma_{ij}} D_{ijkl}\dot{\varepsilon}_{kl} = 2G\begin{bmatrix} \dot{\varepsilon}_{11}-\dot{\varepsilon}_{33} \\ \dot{\varepsilon}_{22}-\dot{\varepsilon}_{33} \end{bmatrix}. \tag{122}$$

Solution of the equation system (26) then requires for fully plastic loading that

$$\begin{aligned} \dot{\lambda}^1 &= \frac{2G}{(A^{11})^2-(A^{12})^2}[A^{11}\dot{\varepsilon}_{11}-A^{12}\dot{\varepsilon}_{22}-(A^{11}-A^{12})\dot{\varepsilon}_{33}] > 0 \\ \dot{\lambda}^2 &= \frac{2G}{(A^{11})^2-(A^{12})^2}[-A^{12}\dot{\varepsilon}_{11}+A^{11}\dot{\varepsilon}_{22}-(A^{11}-A^{12})\dot{\varepsilon}_{33}] > 0. \end{aligned} \tag{123}$$

Alternatively, when a stress driven format is possible, $\dot{f}^I = 0$ yields with (21), (118) and (120) that

$$\dot{\lambda}^1 = \frac{\dot{\sigma}_1-\dot{\sigma}_3}{k^2\psi_{11}} > 0; \quad \dot{\lambda}^2 = \frac{\dot{\sigma}_2-\dot{\sigma}_3}{k^2\psi_{11}} > 0. \tag{124}$$

From (123) or (124), it is apparent that a wide range of strain rates (or stress rates) gives rise to fully plastic loading.

11. TRESCA YIELD FUNCTION—VON MISES POTENTIAL FUNCTION

In the final example, we consider non-associated plasticity with yielding described by Tresca's yield criterion and the potential function in terms of von Mises's criterion. Excluding corner loading, this plasticity model has been used in the past, cf. for instance Mendelsohn (1968) p. 108. More generally, in soil and rock mechanics a Coulumb yield criterion in combination with a Drucker-Prager potential function may be considered as a prototype for a relevant non-associated plasticity formulation and the model considered here may be viewed as a special case of this more general model.

The two yield functions are again described by (105), but as a potential function, we now have

$$g = (\tfrac{3}{2}s_{kl}s_{kl})^{1/2}+c_\alpha A_\alpha-\sigma_{yo}. \tag{125}$$

Here s_{ij} is the deviatoric stress tensor and c_α denotes some constants. It appears that $F_{max} = 2$ and $G_{max} = 1$. Hemholtz's free energy is again taken in the form of (107), which leads to

$$\sigma_{ij} = D_{ijkl}(\varepsilon_{kl}-\varepsilon^p_{kl}); \quad A_\alpha = -\frac{\partial\psi^p}{\partial\kappa_\alpha}. \tag{126}$$

As before, we have two conjugated forces A_1 and A_2 and thereby two internal variables κ_1 and κ_2. Moreover

$$\frac{\partial f'}{\partial A_\alpha} = k_{I\alpha}; \quad \frac{\partial g}{\partial A_\alpha} = c_\alpha \tag{127}$$

and, as before, we use the notation

$$\psi_{\alpha\beta} = \frac{\partial^2 \psi^p}{\partial \kappa_\alpha \partial \kappa_\beta}. \tag{128}$$

The evolution laws (15) then become

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{3s_{ij}}{2\sigma_{eff}}; \quad \dot{\kappa}_\alpha = \dot{\lambda}c_\alpha \tag{129}$$

where $\sigma_{eff} = (3s_{kl}s_{kl}/2)^{1/2}$. From (22), (127) and (128) it follows that

$$H^{I1} = k_{I\alpha}\psi_{\alpha\beta}c_\beta. \tag{130}$$

It seems natural to assume that the two yield surfaces expand by the same amount, irrespective of whether we have loading only along one of the smooth yield surfaces or corner loading, cf. Fig. 4. This is achieved by assuming that

$$k_{I\alpha} = \begin{bmatrix} k & k \\ k & k \end{bmatrix}. \tag{131}$$

With (106), this implies that corner loading occurs when

$$\sigma_1 = \sigma_2 (> \sigma_3). \tag{132}$$

From (130) and (131) it appears that

$$H^{I1} = \begin{bmatrix} H \\ H \end{bmatrix} \quad \text{where} \quad H = k(\psi_{11}c_1 + \psi_{12}(c_1 + c_2) + \psi_{22}c_2) \tag{133}$$

which for $H \neq 0$ fulfills rank $(H^I) = G_{max} = 1$, i.e. a stress driven formulation is possible, cf. (63). Hardening corresponds to $H > 0$, softening to $H < 0$; moreover perfect plasticity and the existence of a limit point coincide and occur when $H = 0$.

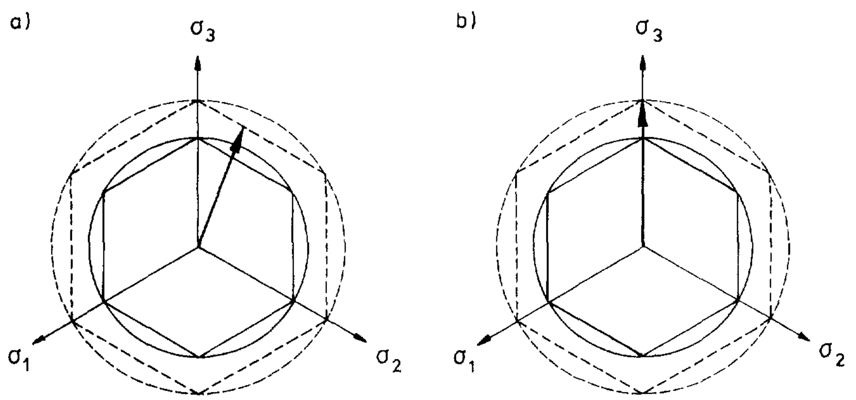

Fig. 4. Deviatoric plane. Same isotropic hardening of the two yield surfaces; a) loading along a smooth yield surface; b) corner loading.

Let us evaluate the conditions for corner loading in more detail. Due to (132) we have $\sigma_{eff} = (3s_{kl}s_{kl}/2)^{1/2} = 3s_{11} = s_{11} - s_{33}$. For isotropic elasticity, where D_{ijkl} is given by (121), we then find with (81c), (120) and (125) that

$$M^{I1} = \begin{bmatrix} 3G \\ 3G \end{bmatrix}. \tag{134}$$

From (25) and (133) it then follows that

$$A^{I1} = \begin{bmatrix} A \\ A \end{bmatrix} \quad \text{where} \quad A = H + 3G. \tag{135}$$

The equation system (26) corresponds to two equations and one unknown. Using (122b) this leads to

$$\dot{\lambda} = \frac{2G}{A}(\dot{\varepsilon}_{11} - \dot{\varepsilon}_{33}) = \frac{2G}{A}(\dot{\varepsilon}_{22} - \dot{\varepsilon}_{33}) > 0 \tag{136}$$

for plastic loading. Since the two expressions for $\dot{\lambda}$ should concide, it is concluded that

$$\dot{\varepsilon}_{11} = \dot{\varepsilon}_{22}(> \dot{\varepsilon}_{33}). \tag{137}$$

Alternatively, when $H \neq 0$, a stress driven format is possible and $\dot{f}^I = 0$ with (21), (120) and (133) shows that

$$\dot{\lambda} = \frac{1}{H}(\dot{\sigma}_{11} - \dot{\sigma}_{33}) = \frac{1}{H}(\dot{\sigma}_{22} - \dot{\sigma}_{33}) > 0 \tag{138}$$

i.e.,

$$\dot{\sigma}_{11} = \dot{\sigma}_{22} \quad \text{and} \quad \dot{\sigma}_{11} > \dot{\sigma}_{33} \quad \text{if } H > 0. \tag{139}$$

It is evident that corner loading is only possible under very special conditions. This is even more pronounced when a comparison is made with the Tresca-Tresca material, cf. (123) and (124). In the latter case, a wide range of strain rates (or stress rates) may lead to corner loading in evident contrast to the present situation.

This leads to another interesting observation. Irrespective of whether we have corner loading or not, the evolution laws are given by (129). Suppose we have corner loading and suppose that we artificially change the stress state slightly so that only one smooth yield surface is active. In the limit, the expression for $\dot{\lambda}$ will still be one of the two expressions in (136). From a numerical point of view this implies that if corner loading is encountered, the state may be changed by an infinitely small amount and everything may be calculated as if only one yield function is activated. In the limit, this procedure provides the exact answer.

Let us finally investigate the uniqueness properties of the model in question. Referring to (102), loss of uniqueness requires that $V^{11} = 0$, where $V^{\Psi\Theta}$ is defined by (90), i.e.,

$$4\underline{F}^{IJ}A^{J1}A^{I1} - 2\underline{F}^{IJ}A^{J1}\underline{A}^{1S}M^{S1}A^{I1} - G^{11} - 2M^{K1}\underline{F}^{KJ}A^{J1} + M^{K1}\underline{F}^{KJ}A^{J1}\underline{A}^{1S}M^{S1} = 0. \tag{140}$$

The quantities A^{I1} and M^{S1} are given by (134) and (135). From (81a), (120) and (121), we obtain

$$F^{IJ} = 2G\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{i.e.,} \quad \underline{F}^{IJ} = \frac{1}{6G}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{141}$$

Moreover, (81b), (121) and (125) imply

$$G^{11} = 3G. \tag{142}$$

In (140), the quantity $\underline{A}^{1S}M^{S1}$ appears and with (134) we get

$$\underline{A}^{1S}M^{S1} = 3G(\underline{A}^{11}+\underline{A}^{12}).$$

The problem is that the left inverse $\underline{A}^{1S}$ is not uniquely defined. However, from (30) and (135) it follows that

$$1 = \underline{A}^{1S}A^{S1} = A(\underline{A}^{11}+\underline{A}^{12}).$$

It then appears that

$$\underline{A}^{1S}M^{S1} = \frac{3G}{A}. \tag{143}$$

Use of (134), (135) and (141)–(143) in (140) give, after some algebra,

$$\frac{4A}{3G}(A-3G) = 0 \tag{144}$$

and since $A = H+3G$ we find that the critical conditions are $A = 0$, i.e., $H = -3G$ and $A-3G = 0$, i.e. $H = 0$. The latter condition is the most critical and it is concluded that

$$H > 0 \Rightarrow \text{uniqueness} \tag{145}$$

i.e., uniqueness is ensured for hardening plasticity even though a non-associated formulation is considered.

REFERENCES

Ayres, F. (1962). *Theory and Problems of Matrices*, Schaum Pub. Co., New York.

Budiansky, B. and Wu, T. T. (1962). Theoretical prediction of plastic strains of polycrystals. In *Proc. 4th U.S. Nat. Congr. Appl. Mech.*, New York, pp. 1175–1185.

Drucker, D. C. (1951). A more fundamental approach to plastic stress strain relations. In *Proc. 1st U.S. Natl. Congr. Appl. Mech.*, Chicago, pp. 487–491.

Hill, R. (1966). Generalized constitutive relations for incremental deformation of metal crystals by multislip. *J. Mech. Phys. Solids* **14**, 95–102.

Koiter, W. T. (1953). Stress–strain relations, uniqueness and variational theorems for elasto-plastic material with a singular yield surface. *Quart. Appl. Math.* **11**, 350–354.

Koiter, W. T. (1960). General theorems for elasto-plastic solids. In *Prog. Solid Mech.* (eds I. N. Sneddon and R. Hill). North-Holland Publ. Company, pp. 165–211.

Lubliner, J. (1972). On the thermodynamic foundations of non-linear solid mechanics. *Int. J. Non-Linear Mechanics* **7**, 237–254.

Luenberger, D. G. (1984). *Linear and Nonlinear Programming*. 2nd edn., Addison-Wesley Pub. Comp.

Maier, G. and Hueckel, T. (1979). Nonassociated and coupled flow rules for elastoplasticity for rock-like material. *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.* **16**, 77–92.

Mandel, J. (1965). Generalisation de la theorie de plasticite de W. T. Koiter. *Int. J. Solids Structures* **1**, 273–295.

Mendelsohn, A. (1968). *Plasticity: Theory and Application*. The MacMillan Comp.

Moreau, J. J. (1974). On unilateral constraints, friction and plasticity. In *New Variational Techniques in Mathematical Physics* (eds G. Capriz and G. Stampacchia), Edizion: Cremonese, pp. 173–322.

Sewell, M. J. (1973). A yield-surface corner lowers the buckling stress of an elastic-plastic plate under compression. *J. Mech. Phys. Solids* **21**, 19–45.

Sewell, M. J. (1974). A plastic flow rule at yield vertex. *J. Mech. Phys. Solids* **22,** 469–490.

Simo, J. C., Kennedy, J. G. and Govindjee, S. (1988). Non-smooth multisurface plasticity and viscoplasticity. Loading/unloading conditions and numerical algorithms. *Int. J. Num. Meth. Engng* **26,** 2161–2185.

Taylor, G. I. (1938). Plastic strain in metals. *J. Inst. Metals* **62,** 307–324.

Warner, W. H. and Handelman, G. H. (1956). A modified incremental strain law for work-hardening materials. *J. Mech. & Appl. Math.* **9,** 279.